Indian Statistical Institute, Bangalore Centre
M.Math. (I Year) : 2015-2016
Semester I : Semestral Examination
Measure Theoretic Probability

12.11.2013 Time: 3 hours. Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. (20 marks) Let $f(\cdot)$ be a non-negative measurable function on a $\sigma-$finite measure space $(\Omega, \mathcal{B}, \mu)$. Show that $f(\cdot) = 0,\ \mu-$a.e. if and only if $\int_\Omega f(\omega) d\mu(\omega) = 0$.

2. (15 marks) Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. Let $\mathcal{A} \subseteq \mathcal{B}$ be a sub-$\sigma-$algebra. Let X be real valued random variable having finite expectation. Show that there is an $\mathcal{A}-$measurable and integrable random variable Y, such that $\int_A X(\omega) d\mathbb{P}(\omega) = \int_A Y(\omega) d\mathbb{P}(\omega)$ for all $A \in \mathcal{A}$, and that Y is unique $\mathbb{P}$-a.e.

3. (15 marks) Let $\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}^2)$ denote respectively the Borel $\sigma-$algebras of $\mathbb{R}, \mathbb{R}^2$. Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, where the right-hand side is the smallest $\sigma-$algebra containing all measurable rectangles.

4. (15 + 10 + 10 = 35 marks) Let $X,\ X_n, n = 1, 2, \cdots$ be real valued random variables defined on a probability space $(\Omega, \mathcal{B}, P)$.

   (i) If $X_n \to X$ in probability, show that there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k} \to X\ P-$a.e. as $k \to \infty$.

   (ii) Let $X_n \to X$ in probability. Suppose there is a random variable Y which is integrable with respect to P, such that $|X_n(\omega)| \le Y(\omega),\ \omega \in \Omega,\ n \ge 1$. Show that X is integrable with respect to P, and that

   $$\int_\Omega X(\omega) dP(\omega) = \lim_{n\to\infty} \int_\Omega X_n(\omega) dP(\omega).$$

   (iii) If $X_n \Rightarrow b$ as $n \to \infty$, where $b \in \mathbb{R}$ is a constant, show that $X_n \to b$ in probability.

5. ($8 + 12 = 20$ marks) (i) Let $X,\ Y$ be independent real valued random variables. Express the characteristic function of $X+Y$ in terms of the characteristic functions of X and Y.

   (ii) For $n = 1, 2, \cdots$ let $Z_n = X + Y_n$, where X and Y_n are independent real valued random variables, and Y_n has $N(0, \frac{1}{n})$ distribution. Does $\{Z_n\}$ converge in distribution? Justify.